

$K_{s,t}$ -saturated bipartite graphs

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Abstract

An n -by- n bipartite graph is H -saturated if the addition of any missing edge between its two parts creates a new copy of H . In 1964, Erdős, Hajnal and Moon made a conjecture on the minimum number of edges in a $K_{s,s}$ -saturated bipartite graph. This conjecture was proved independently by Wessel and Bollobás in a more general, but ordered, setting: they showed that the minimum number of edges in a $K_{(s,t)}$ -saturated bipartite graph is $n^2 - (n - s + 1)(n - t + 1)$, where $K_{(s,t)}$ is the “ordered” complete bipartite graph with s vertices in the first color class and t vertices in the second. However, the very natural question of determining the minimum number of edges in the unordered $K_{s,t}$ -saturated case remained unsolved. This problem was considered recently by Moshkovitz and Shapira who also conjectured what its answer should be. In this short paper we give an asymptotically tight bound on the minimum number of edges in a $K_{s,t}$ -saturated bipartite graph, which is only smaller by an additive constant than the conjecture of Moshkovitz and Shapira. We also prove their conjecture for $K_{2,3}$ -saturation, which was the first open case.

1 Introduction

For two graphs G and H , G is said to be H -saturated if it contains no copy of H as a subgraph, but the addition of any edge missing from G creates a copy of H in G . The saturation number $\text{sat}(n, H)$ is defined as the minimum number of edges in an H -saturated graph on n vertices. Notice that the problem of finding the saturation number for H is, in some sense, the dual of the classical Turán problem.

Probably the most natural setup of this problem is when we choose H to be a fixed complete graph K_s . This was first studied by Zykov [10] in the 1940’s, and later by Erdős, Hajnal and Moon [5] in 1964. They proved that $\text{sat}(n, K_s) = (s - 2)n - \binom{s-1}{2}$. Here the upper bound comes from the K_s -saturated graph that has $s - 2$ vertices connected to all other vertices. Later, the closely related notion of weak saturation was introduced by Bollobás [4]. A graph G is weakly H -saturated if it is possible to add back the missing edges of G one by one in some order, so that each addition creates a new copy of H . Trivially, if G is H -saturated, then any order satisfies this property, hence G is also weakly H -saturated. Let $w\text{-sat}(n, K_s)$ be the minimum number of edges in an n -vertex graph that is weakly K_s -saturated. We then have $w\text{-sat}(n, K_s) \leq \text{sat}(n, K_s)$. Somewhat surprisingly, one can prove using algebraic techniques (see e.g. [7]) that these two functions are actually equal. On

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the other hand, the extremal graphs for these problems are not the same and already for $s = 3$ there are weakly K_3 -saturated graphs (i.e. trees) which are not K_3 -saturated.

The paper by Erdős, Hajnal and Moon also introduced the bipartite saturation problem, where we are looking for the minimum number of edges $\text{sat}(K_{n,n}, H)$ in an H -free n -by- n bipartite graph, such that adding any missing edge between the two color classes creates a new copy of H . (Of course, this definition is only meaningful if H is also bipartite.) They conjectured that $\text{sat}(K_{n,n}, K_{s,s}) = n^2 - (n - s + 1)^2$. Once again, this is seen to be tight by selecting $s - 1$ vertices on each side of the bipartite graph and connecting them to every vertex on the opposite side. In the bipartite setting, one can impose an additional restriction on the problem by ordering the two vertex classes of H and requesting that each missing edge create an H respecting the order: the first class of H lies in the first class of G . For example, let $K_{(s,t)}$ be the complete “ordered” s -by- t bipartite graph with s vertices in the first class and t vertices in the second, then a bipartite graph G is $K_{(s,t)}$ -saturated if each missing edge creates a $K_{s,t}$ with the s -vertex class lying in the first class of G . Indeed, the conjecture of Erdős, Hajnal and Moon was independently confirmed by Wessel [9] and Bollobás [3] a few years later as the special case of the following result: $\text{sat}(K_{(n,n)}, K_{(s,t)}) = n^2 - (n - s + 1)(n - t + 1)$. This was further generalized in the 80s by Alon [1] to complete k -uniform hypergraphs in a k -partite setting using algebraic tools. Alon showed that the saturation and weak saturation bounds are the same in this case as well. For a more detailed discussion of H -saturation in general, we refer the reader to the survey [6] by Faudree, Faudree and Schmitt.

In this paper we study the unordered case of bipartite saturation. Although this is arguably the most natural setting for the bipartite problem, it did not receive any attention until very recently in [2, 8]. Moshkovitz and Shapira [8] studied the unordered weak saturation number of $K_{s,t}$, $s \leq t$, and showed that $w\text{-sat}(K_{n,n}, K_{s,t}) = (2s - 2 + o(1))n$. Note that, surprisingly, it is much smaller than the corresponding ordered saturation number and only depends on the size of the smaller part. One might think that a similar gap exists for saturation numbers as well. Moshkovitz and Shapira [8] conjectured that this is not the case, and that ordered and unordered bipartite saturation numbers differ only by an additive constant. More precisely, they made the following conjecture and constructed an example showing that, if true, this bound is tight.

Conjecture 1.1. *Let $1 \leq s \leq t$ be integers. Then there is an n_0 such that if $n \geq n_0$ and G is a $K_{s,t}$ -saturated n -by- n bipartite graph, then G contains at least $(s + t - 2)n - \left\lfloor \left(\frac{s+t-2}{2}\right)^2 \right\rfloor$ edges.*

In this short paper we prove the following result which confirms the above conjecture up to a small additive constant.

Theorem 1.2. *Let $1 \leq s \leq t$ be fixed and $n \geq t$. Then*

$$\text{sat}(K_{n,n}, K_{s,t}) \geq (s + t - 2)n - (s + t - 2)^2.$$

The proof is presented in Section 2. In Section 3, we show that if the conjecture is true, it has many extremal examples. Finally, in Section 4, we prove Conjecture 1.1 in the first open case of $K_{2,3}$ -saturation.

2 Lower bounds on the saturation number

Let G be a bipartite graph with vertex class U and U' of size n . Assume $1 \leq s \leq t \leq n$ and suppose G is $K_{s,t}$ -saturated, i.e. each missing edge between U and U' creates a new $K_{(s,t)}$ or a new $K_{(t,s)}$ when added to G . Here $K_{(a,b)}$ refers to a complete bipartite graph with a vertices in U and b vertices in U' .

Let us start with the following, easy special case of Theorem 1.2.

Proposition 2.1. *Suppose a $K_{s,t}$ -saturated graph has minimum degree $\delta < t - 1$. Then it contains at least $n(t + s - 2) - (s + t - 2)^2$ edges.*

Proof. The $K_{s,t}$ -saturated property ensures that each vertex has at least $s - 1$ neighbors, so we actually have $s - 1 \leq \delta < t - 1$. Let u_0 be a vertex of degree δ ; we may assume that $u_0 \in U$. Then adding any missing edge $u_0 u'$ to G (where $u' \in U' - N(u_0)$) should create a new $K_{(t,s)}$ because it cannot create a $K_{(s,t)}$. For such a u' , let $S_{u'} \subseteq U$ be the $t - 1$ vertices other than u_0 in the t -class of this $K_{(t,s)}$, and define $V \subseteq U$ to be the union of these $S_{u'}$. Then all vertices in V have at least $s - 1$ neighbors in $N(u_0)$ and all vertices in $U' - N(u_0)$ have at least $t - 1$ neighbors in V . Now we can count the number of edges in G as follows:

$$\begin{aligned} e(U, U') &= e(V, N(u_0)) + e(V, U' - N(u_0)) + e(U - V, U') \\ &\geq (s - 1)|V| + (t - 1)(n - |N(u_0)|) + \delta(n - |V|) \\ &\geq (s - 1)|V| + (t - 1)(n - t + 2) + (s - 1)(n - |V|) \\ &\geq n(t + s - 2) - (t - 1)(t - 2) \\ &\geq n(t + s - 2) - (s + t - 2)^2. \end{aligned}$$

□

The case when $\delta \geq t - 1$ is considerably more complicated. We introduce the following structure to count the edges of G (see Figure 1). The *core* of this structure is a set $\tilde{A}_0 = A_0 \cup A'_0$ with $A_0 \subseteq U$ and $A'_0 \subseteq U'$ satisfying the following technical property:

- there are vertices $u_0 \in A_0$ and $u'_0 \in A'_0$ such that their neighborhoods are also contained in the core.

Next, we build the shell around the core: starting with $\tilde{A} = \tilde{A}_0$, we iteratively add any vertex v to \tilde{A} that has at least $t - 1$ neighbors in it. In other words, $\tilde{A} = A \cup A'$ is the smallest set containing \tilde{A}_0 such that any vertex $v \in G - \tilde{A}$ has fewer than $t - 1$ neighbors in \tilde{A} . Here $A_0 \subseteq A \subseteq U$ and $A'_0 \subseteq A' \subseteq U'$. We use the variables $x_0 = |A_0|$, $x'_0 = |A'_0|$, $x = |A|$ and $x' = |A'|$ to denote the sizes of the corresponding sets. Obviously $x_0 \leq x$ and $x'_0 \leq x'$.

The following, rather scary, lemma is the key to our lower bounds on the saturation numbers. It shows that we can find about $n(s + t - 2)$ edges in a $K_{s,t}$ -saturated graph, provided we have a small enough core.

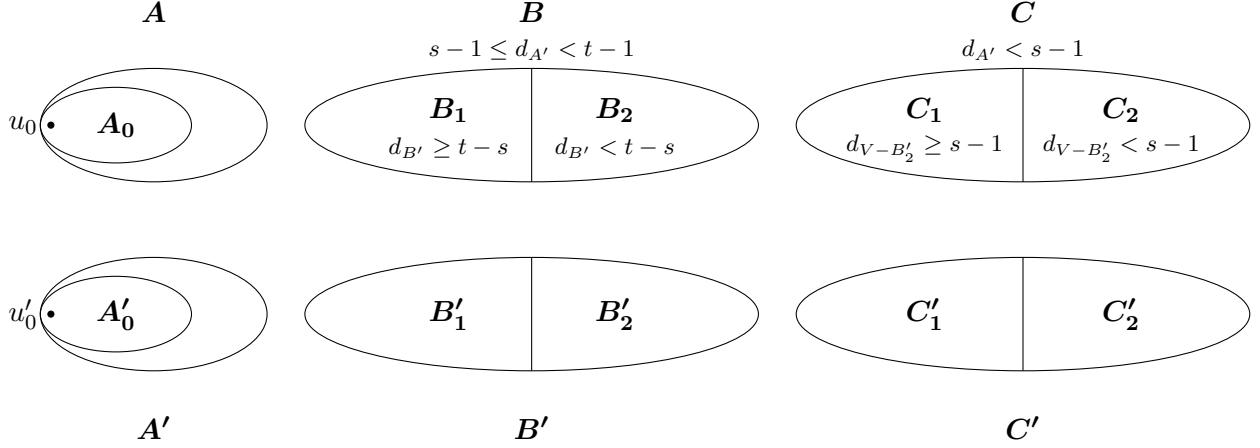


Figure 1: the structure for counting the edges

Lemma 2.2. *Assuming $\delta \geq t - 1$, suppose the core spans $e = e(A_0, A'_0)$ edges. Then G has at least*

$$n(s + t - 2) - (x_0 + x'_0)(t - 1) - \left\lfloor \frac{(s - 1)^2}{4} \right\rfloor + e + \min\{(t - s)x, (t - s)x'\}$$

edges.

Proof. By the construction of \tilde{A} , we know that it spans at least $e + (t - 1)(x + x' - x_0 - x'_0)$ edges. Indeed, each vertex we added to the shell brings at least $t - 1$ new edges. The idea is to count $t - 1$ edges from the remaining vertices on one side of the graph, say $U' - A'$, and then to find $s - 1$ new (yet uncounted) edges from the other side, $U - A$. Of course, if a vertex in $U - A$ has at least $s - 1$ neighbors in A' , then these edges are guaranteed to be new.

So let us continue with our definition of the structure. We know that any vertex in $U - A$ has fewer than $t - 1$ neighbors in A' . We break this set into two parts by defining B to be the set of vertices in $U - A$ having at least $s - 1$ neighbors in A' , and C to be those having fewer than $s - 1$ neighbors in A' . Similarly we break $U' - A'$ into two sets B' and C' based on the size of the neighborhood in A . We need to break B further into two parts B_1 and B_2 , by defining B_1 to be the set of vertices having at least $t - s$ neighbors in B' . Similarly, let B'_1 be the set of vertices in B' having at least $t - s$ neighbors in B (see Figure 1).

Note that any vertex in B'_1 already has $t - 1$ neighbors in $A \cup B$ (at least $s - 1$ in A and at least $t - s$ in B), but this is not necessarily true for B'_2 . This, together with our strategy to find $s - 1$ new edges from the vertices in C motivates our last partitioning: We now break C into two parts C_1 and C_2 , where C_1 is the set of those vertices in C which have at least $s - 1$ neighbors outside B'_2 , and $C_2 = C - C_1$. We similarly define $C'_1 = \{v \in C' : |N(v) - B_2| \geq s - 1\}$, where $N(v)$ is the neighborhood of v , and $C'_2 = C' - C'_1$.

An observation here, which will prove to be crucial when counting the edges, is that C_2 and C'_2 span a complete bipartite graph. Indeed, suppose there is a missing edge vv' in G , with $v \in C_2$ and $v' \in C'_2$. Adding this edge creates a $K_{(s,t)}$ or a $K_{(t,s)}$, suppose it is a $K_{(s,t)}$. Then v' is connected

to all the $s - 1$ vertices other than v in the s -vertex class of this $K_{(s,t)}$. But v' is in C'_2 , so it has at most $s - 2$ neighbors outside B_2 , consequently there is a vertex $w \in B_2$ in the s -class. Similarly, using that v is in C_2 , we find at least $t - s$ vertices of the t -class in B'_2 . But then $w \in B_2$ has at least $t - s$ neighbors in $B'_2 \subseteq B'$, which contradicts the definition of B_2 . The same argument leads to a contradiction if the edge creates a $K_{(t,s)}$, hence we can conclude that there is no missing edge between C_2 and C'_2 .

On another note, observe that adding the edge u_0v' , where u_0 is the vertex in A_0 defined in the property of the core and v' is any vertex in C' , cannot create a $K_{(s,t)}$. Indeed, if it created a $K_{(s,t)}$, then all the vertices of the t -class except v' are neighbors of u_0 , so they are sitting in the core, A'_0 . This means that each vertex in the s -class is connected to at least $t - 1$ vertices in the core, hence the whole s -class is in A . But then v' has at least $s - 1$ neighbors in A , contradicting $v' \in C'$. So we see that adding u_0v' creates a $K_{(t,s)}$. Then, all the vertices of the s -class of this copy of $K_{(t,s)}$ except v' are in A'_0 , therefore the vertices of the t -class have at least $s - 1$ neighbors in A' . Hence all of them are in $A \cup B$, implying that every $v' \in C'$ has at least $t - 1$ neighbors in $A \cup B$. The same argument shows that each $v \in C$ has at least $t - 1$ neighbors in $A' \cup B'$.

Lemma 2.2 will now follow from the following claim, possibly applied to the graph with the two vertex classes switched.

Claim 2.3. *Assuming $\delta \geq t - 1$, suppose $|C_2| \leq |C'_2|$. Then*

$$e(U, U') \geq n(s + t - 2) - (x_0 + x'_0)(t - 1) - \left\lfloor \frac{(s - 1)^2}{4} \right\rfloor + e + (t - s)x.$$

Proof. Let $y = |C_2|$ and $y' = |C'_2|$, and let us count the edges in G . We noted above that each vertex in B'_1 has at least $t - 1$ neighbors in $A \cup B$, so $e(A \cup B, B'_1) \geq (t - 1)|B'_1|$. By assumption, each vertex in B'_2 has degree at least $t - 1$, hence $e(A \cup B \cup C, B'_2) \geq (t - 1)|B'_2|$. We have also shown that each vertex in C' has at least $t - 1$ neighbors in $A \cup B$, so $e(A \cup B, C') \geq (t - 1)|C'|$. This so far means that

$$e(A \cup B, B'_1) + e(A \cup B \cup C, B'_2) + e(A \cup B, C') \geq (t - 1)(n - x'). \quad (1)$$

Now look at what we have left from the other side: By definition, any vertex in B has at least $s - 1$ neighbors in A' , so $e(B, A') \geq (s - 1)|B|$. We also defined C_1 so that its vertices have at least $s - 1$ neighbors outside B'_2 , this gives $e(C_1, A' \cup B'_1 \cup C') \geq (s - 1)|C_1|$. As we noted above, the vertices of C_2 are all connected to the vertices of C'_2 , so $e(C_2, C'_2) = yy'$. Using the fact that $y(s - 1 - y) \leq \left\lfloor \frac{(s-1)^2}{4} \right\rfloor$ (y is an integer), we get that

$$\begin{aligned} e(B, A') + e(C_1, A' \cup B'_1 \cup C') + e(C_2, C'_2) &\geq (s - 1)(n - x - y) + yy' \\ &\geq (s - 1)(n - x) - (s - 1)y + y^2 \\ &\geq (s - 1)(n - x) - \left\lfloor \frac{(s - 1)^2}{4} \right\rfloor. \end{aligned} \quad (2)$$

We have also seen that $e(A, A')$ is at least $e + (t - 1)(x + x' - x_0 - x'_0)$.

It is easy to check that we never counted an edge more than once above, hence

$$\begin{aligned} e(U, U') &\geq (t-1)(n-x') + (s-1)(n-x) + (t-1)(x+x'-x_0-x'_0) + e - \left\lfloor \frac{(s-1)^2}{4} \right\rfloor \\ &= n(t+s-2) + (t-s)x - (x_0+x'_0)(t-1) + e - \left\lfloor \frac{(s-1)^2}{4} \right\rfloor, \end{aligned} \quad (3)$$

what we wanted to show. \square

We state the following immediate corollary of this claim, which we need in Section 4.

Corollary 2.4. *If we have equality in Claim 2.3, then the following statements hold:*

- any vertex in $B'_1 \cup C'$ has exactly $t-1$ neighbors in $A \cup B$,
- any vertex in B has exactly $s-1$ neighbors in A' ,
- the vertices in C_1 have exactly $s-1$ neighbors outside B'_2 , and
- $y(s-1) - yy' = \lfloor (s-1)^2/4 \rfloor$.

Now we are ready to prove our general theorem, which is tight up to an additive constant. Let us emphasize, however, that since our methods do not give the exact result, we will not make any effort to optimize the constant error term.

Theorem 2.5. *If $G = (U, U', E)$ is a $K_{s,t}$ -saturated bipartite graph with n vertices on each side, then it contains at least $(s+t-2)n - (s+t-2)^2$ edges.*

Proof. Following Lemma 2.2, our plan is to find an appropriate core.

By Proposition 2.1, we may assume that the minimum degree of our graph is at least $t-1$. Suppose for contradiction that G contains fewer than $(s+t-2)n - (s+t-2)^2$ edges. Then there is a vertex $u_0 \in U$ of degree at most $s+t-3$. Moreover, there is a non-adjacent vertex $u'_0 \in U' - N(u_0)$ of degree at most $s+t-3$ as well, since otherwise the number of edges in G would be at least $(n - (s+t-3))(s+t-2) > (s+t-2)n - (s+t-2)^2$, contradicting our assumption. Set $A_0 = \{u_0\} \cup N(u'_0)$ and $A'_0 = \{u'_0\} \cup N(u_0)$, and define $\tilde{A}_0 = A_0 \cup A'_0$ to be the core.

Using the above notation, we see that $x_0 = |A_0| = 1 + |N(u'_0)| \leq s+t-2$ and $x'_0 = |A'_0| = 1 + |N(u_0)| \leq s+t-2$. Since u_0 and u'_0 are not adjacent, we can add the edge $u_0u'_0$ to create a new $K_{s,t}$. Notice that all the vertices of this $K_{s,t}$ are adjacent to either u_0 or u'_0 , hence they all lie in the core. Consequently, the core spans $e = e(A_0, A'_0) \geq st-1$ edges. Now applying Lemma 2.2 we get

$$\begin{aligned} e(U, U') &\geq n(s+t-2) - (x_0+x'_0)(t-1) + \min\{(t-s)x, (t-s)x'\} - \left\lfloor \frac{(s-1)^2}{4} \right\rfloor + e \\ &\geq n(s+t-2) - (x_0+x'_0)(t-1) + \min\{(t-s)x_0, (t-s)x'_0\} - \left\lfloor \frac{(s-1)^2}{4} \right\rfloor + st-1 \\ &\geq n(s+t-2) - (s+t-2)^2 + st-1 - \left\lfloor \frac{(s-1)^2}{4} \right\rfloor \\ &\geq n(s+t-2) - (s+t-2)^2. \end{aligned} \quad (4)$$

This contradicts the assumption, thus proving the theorem. \square

3 Extremal graphs

As we mentioned in the introduction, Moshkovitz and Shapira [8] constructed a $K_{s,t}$ -saturated n -by- n bipartite graph showing that the bound of the Conjecture 1.1, if true, is tight. It appears that this example is not unique. In this section we describe a general family of such graphs which contains the example by Moshkovitz and Shapira as a special case (when $l = 1$).

Example. As usual, we denote the two sides of the bipartite graph by U and U' , where $|U| = |U'| = n$. Let us break each class into two major parts: $U = V \cup W$ and $U' = V' \cup W'$, where $|V| = |V'| = \lfloor \frac{t+s-2}{2} \rfloor$ (assume n is large enough). Suppose W and W' are further broken into some parts W_1, \dots, W_l and W'_1, \dots, W'_l where $|W_i| = |W'_i| \geq t-s$ for all i . The construction of an extremal graph G goes as follows.

First include in G all the edges between V and V' , making it a complete bipartite graph. Also, for every i , choose the edges between W_i and W'_i to span an arbitrary $(t-s)$ -regular graph. It remains to describe the edges going between different type of classes.

We do not include any edge between W_i and W'_j for any $i \neq j$. Instead, choose arbitrary sets $S' \subseteq V'$ and $S_1, \dots, S_l \subseteq V$ of size $s-1$, and take all edges going between W_i and S' as well as the edges between S_i and W'_i , for all i . A straightforward computation shows that the number of edges in this G is exactly the number in the conjecture. We claim that G is $K_{s,t}$ -saturated.

Let us see what happens when we add a missing edge uu' to G . If $u' \in W'$, i.e. $u' \in W'_i$ for some i , then let N be the set of its $t-s$ neighbors in W_i . Since $u \in U - N - S_i$, the set $S_i \cup \{u\} \cup N \cup S' \cup \{u'\}$ then forms a $K_{(t,s)}$. On the other hand, if $u' \in V'$, then $u \in W_i$ for some i . Let N' be the set of the $t-s$ neighbors of u in W'_i , then $u' \in U' - N' - S'$ and hence the set $S_i \cup \{u\} \cup S' \cup N' \cup \{u'\}$ forms a $K_{(s,t)}$. This proves the saturation property.

The asymmetric structure of the above example comes from the relaxation of the $l = 1$ case, which corresponds to the construction of Moshkovitz and Shapira. When all the vertices in W' are connected to the same subset of V of size $s-1$, adding an edge between W and W' creates both a $K_{(s,t)}$ and a $K_{(t,s)}$. Our example exploits the freedom we had in choosing the edges between W' and V . In our case, when $l > 1$, adding an edge between W_i and W'_j with $S_i \neq S_j$ creates only a $K_{(t,s)}$. The existence of such asymmetric examples provides further difficulties in proving an exact result.

4 The $K_{2,3}$ case

For $t = s$, Conjecture 1.1 trivially follows from the ordered result by Bollobás [3]. The other extreme is also easy to handle. When $s = 1$, the $K_{1,t}$ -saturated property merely means that the vertices of degree less than $t-1$ span a complete bipartite graph. Then it is a simple exercise to show (see [8]) that the conjecture holds in this case as well.

Thus the first open case is $s = 2$ and $t = 3$, where the conjecture asserts that any $K_{2,3}$ -saturated graph contains at least $3n - 2$ edges. We note that there are many saturated graphs on $3n - 2$ edges. In fact, there are many such examples which are even $K_{(2,3)}$ -saturated: Just take a vertex $v' \in U'$ that is connected to everything in U , and make sure that every other vertex in U' has degree 2.

In this section we prove the matching lower bound. A brief summary of the coming theorem can be phrased as follows. By finding an appropriate core, our techniques from Section 2 easily give a $3n - 3$ lower bound. The rest of the proof is then a series of small structural observations, ultimately ruling out the possibility that a $K_{2,3}$ -saturated graph with $3n - 3$ edges exists.

Theorem 4.1. *If $G = (U, U'; E)$ is a $K_{2,3}$ -saturated bipartite graph with $n \geq 4$ vertices in each part, then it has at least $3n - 2$ edges.*

Proof. As a first step, we show in the spirit of Proposition 2.1 that it is enough to consider graphs of minimum degree 2.

Lemma 4.2. *If G contains fewer than $3n - 2$ edges, then it has minimum degree 2. Moreover, it contains two non-adjacent vertices $u_0 \in U$ and $u'_0 \in U'$ of degree 2.*

Proof. The saturation property ensures that each vertex has at least one neighbor. Suppose there is a vertex u of degree 1 – wlog $u \in U$ – and let $u' \in U'$ be its neighbor. Take any vertex $v' \in U'$ other than u' , then adding the edge uv' cannot create a $K_{(2,3)}$, so it must create a $K_{(3,2)}$, with the 2-vertex class being $\{u', v'\}$. For any such v' , let $U_{v'} \subseteq U$ be the 3-class of this $K_{(3,2)}$, so $U_{v'}$ consists of u and two neighbors of v' . We count the two edges between v' and $U_{v'}$ for each $v' \in U'$, $v' \neq u'$ to get a total of $2n - 2$ different edges.

Now let $X = \cup U_{v'}$, then every vertex in X is connected to u' because each of the above $K_{(3,2)}$'s contains u' . This gives $|X|$ new edges. On the other hand, we still have not encountered any edges touching $U - X$. But since we know that each vertex has at least one neighbor, we surely have at least $n - |X|$ new edges. This is already a total of $3n - 2$ edges in G , contradicting our assumption.

Therefore the minimum degree is at least 2, but in fact it is exactly two, as otherwise we would have at least $3n$ edges in the graph. Let u_0 have degree 2 – we may assume $u_0 \in U$. If every non-adjacent vertex in U' has at least 3 neighbors, then we have $2 \cdot 2 + 3(n - 2) = 3n - 2$ edges incident to U' , again a contradiction. Hence there is a $u'_0 \in U'$ of degree 2 that is not adjacent to u_0 , and we are done. \square

Suppose G is a counterexample to our theorem, and apply Lemma 4.2 to get two non-adjacent vertices u_0 and u'_0 of degree 2. Denote the neighbors of u_0 by $u'_1, u'_2 \in U'$, the neighbors of u'_0 by $u_1, u_2 \in U$, and let $A_0 = \{u_0, u_1, u_2\}$ and $A'_0 = \{u'_0, u'_1, u'_2\}$ be the core of the structure we described at the beginning of Section 2. Using this core we will also construct the sets A , $B = B_1 \cup B_2$, $C = C_1 \cup C_2$ and A' , $B' = B'_1 \cup B'_2$, $C' = C'_1 \cup C'_2$ as defined by the structure.

Assume that $|C_2| \leq |C'_2|$ and apply Claim 2.3 with $s = 2$ and $t = 3$ to the structure of core $\tilde{A}_0 = A_0 \cup A'_0$. These choices for s and t significantly simplify the bound we get from this claim:

$$e(U, U') \geq 3n - 6 \cdot 2 - 0 + e + x = 3n - 12 + e + x.$$

Using that the addition of the edge $u_0 u'_0$ creates a $K_{2,3}$ inside the core (as all neighbors of u_0 and u'_0 are in \tilde{A}_0), it is easy to check that $e = e(A_0, A'_0) \geq 6$. We also know that $x \geq x_0 = 3$, so $e(U, U') \geq 3n - 3$. Then these inequalities together with Corollary 2.4 imply that if $e(U, U') = 3n - 3$ then G satisfies the following five properties:

1. $e = e(A_0, A'_0) = 6$ and $x = |A| = 3$,
2. any vertex in $B'_1 \cup C'$ has exactly 2 neighbors in $A \cup B$,
3. any vertex in B has exactly 1 neighbor in A' ,
4. the vertices in C_1 have exactly 1 neighbor outside B'_2 , and
5. for $y = |C_2|$ and $y' = |C'_2|$ (with $0 \leq y \leq y'$) we have $y(s-1) - yy' = y(1-y') = 0$, so either $y = 0$ or $y = y' = 1$.

The following lemma supplements the fifth property and shows that C_2 must be empty and C'_2 must be non-empty, by taking care of the case $y = y' = 0$ and $y = y' = 1$.

Lemma 4.3. *If $|C_2| = |C'_2|$ then G spans at least $3n - 2$ edges.*

Proof. As G is a counterexample, by Lemma 4.2 it has minimum degree 2. Since $y = y'$, we may apply Claim 2.3 and Corollary 2.4 to G 's "mirror", with U and U' switched, and observe that the five properties hold for this mirror graph as well. Then the first property gives $x = 3$, $x' = 3$ and $e = 6$. So $A = \{u_0, u_1, u_2\}$ and $A' = \{u'_0, u'_1, u'_2\}$ (i.e. any vertex not in the core has at most one neighbor in it), and the core spans 6 edges. By symmetry we can assume that adding the edge $u_0u'_0$ creates a $K_{(2,3)}$ on the set $\{u_0, u_1, u'_0, u'_1, u'_2\}$, so the missing edges are $u_0u'_0$, $u_2u'_1$ and $u_2u'_2$. We also assumed that there is no vertex of degree 1, so u_2 must have some neighbor v' in B' . Note that v' has exactly one neighbor in A , in particular it is not connected to u_1 .

Now let us see what happens when we add the edge u_0v' . We cannot create a $K_{(2,3)}$, because that would use both u'_1 and u'_2 , but their only common neighbor other than u_0 is u_1 (recall that no vertex outside A' can have 2 neighbors in A), which is not connected to v' . So it must be a $K_{(3,2)}$, and it is not using u_2 , as u_2 has no common neighbor with u_0 . But then the $K_{(3,2)}$ contains two neighbors of v' that are not in A , but are connected to a vertex in A' . Then, by definition, these neighbors are in B . So $v' \in B'_1$ has at least two neighbors in B and one in A , and this contradicts the second property. \square

From now on we assume that C_2 is empty and C'_2 is non-empty. Then the fourth property also implies that the vertices in $C = C_1$ have exactly one neighbor outside B'_2 . Moreover, the third property tells us that each vertex in B has exactly one neighbor in A' .

Lemma 4.4. *All vertices in B are connected to the same vertex in A' .*

Proof. Break B into parts based on the neighbor in A' by putting the vertices in B connected to $w' \in A'$ into the set $B_{w'}$. We claim that vertices in different parts do not share common neighbors, or in other words, any vertex $v' \in B' \cup C'$ has all its neighbors in B contained in the same part $B_{w'}$.

Indeed, any vertex in B' has at most one neighbor in B : this is true by definition for the vertices in B'_2 , and follows from the second property for B'_1 (every vertex in B'_1 has a neighbor in A). Now look at the vertices in C' . An easy observation in Lemma 2.2 shows that adding the edge u_0v' for $v' \in C'$ cannot create a $K_{(2,3)}$. So it creates a $K_{(3,2)}$, and this $K_{(3,2)}$ must contain the two neighbors

of v' in B and a neighbor w'_0 of u_0 in A' . Hence both neighbors of v' are in $B_{w'_0}$, establishing the claim.

As we noted above, C' is not empty, so take a vertex $v'_1 \in C'$ and assume that the neighbors of v'_1 in B are in $B_{w'_1}$. We will show that $B = B_{w'_1}$. Suppose not, i.e. there is a $v_1 \in B_{w'_2}$ with $w'_1 \neq w'_2$. Then the edge $v_1 v'_1$ is missing; let us see what happens when we add that edge. We create a $K_{(2,3)}$ or a $K_{(3,2)}$, so in any case there are vertices $v_2 \in U$ and $v'_2 \in U'$ such that $v_1 v'_2$, $v_2 v'_2$ and $v_2 v'_1$ are all edges of G . Here v_2 cannot be in A , as it is connected to $v'_1 \in C'$. It is not in B either, since then one of v'_1 and v'_2 would have neighbors in both $B_{w'_1}$ and $B_{w'_2}$. So $v_2 \in C = C_1$ (since C_2 is empty). Now the fourth property says that v_2 has exactly 1 neighbor outside B'_2 . Since $v'_1 \in C'$, v'_2 must be in B'_2 . But the vertices in B'_2 have no neighbors in B so $v_1 v'_2$ cannot be an edge, giving a contradiction. \square

Note that this lemma implies that one of the two neighbors of u_0 – say u'_1 – is not connected to any vertex in B , and therefore it is adjacent to at least two vertices in $A = A_0$. We also recall that the core only spans six edges. It is time to analyze what happens in the core when we add the edge $u_0 u'_0$. It might create a $K_{(3,2)}$ or a $K_{(2,3)}$, but the obtained graph is inside the core in both cases.

Case 1: $u_0 u'_0$ creates a $K_{(3,2)}$.

If this $K_{(3,2)}$ used u'_2 , then the core of G would contain more than 6 edges: 5 from the $K_{(3,2)}$ and 2 other edges incident to u'_1 , which is impossible. So u'_1 is connected to both u_1 and u_2 , while u'_2 is not connected to any of them. Note, however, that u'_2 is connected to all vertices in B .

Let v be any vertex in $U - A$. When we add the edge vu'_0 to G , we create a $K_{(2,3)}$ or a $K_{(3,2)}$, so there is a vertex v' connected to both v and u_1 or u_2 . Then v' is not in A' , since v is only connected to u'_2 in A' , but both $u_1 u'_2$ and $u_2 u'_2$ are missing. Thus $v' \in B'$ (it has a neighbor in A , so it is not in C'). When we add the edge $u_0 v'$, we cannot create a $K_{(2,3)}$, because that would use both u'_1 and u'_2 , which only share u_0 as their common neighbor. So it creates a $K_{(3,2)}$ using one of u'_1 and u'_2 . It cannot be u'_1 , because then the 3-class of the $K_{(3,2)}$ is exactly A , making v' have 2 neighbors in A . Thus, by definition, $v' \in A'$ which contradicts $v' \in B'$. But it cannot be u'_2 either, because then v' would have two neighbors in B , which together with a neighbor in A that v' must have, contradicts the second property. So this case is impossible.

Case 2: $u_0 u'_0$ creates a $K_{(2,3)}$.

Then one of u_1 and u_2 – say u_1 – is connected to both u'_1 and u'_2 , and the other is connected to neither. But then u'_1 has exactly two neighbors, u_0 and u_1 , and the set $\bar{A} = \{u_0, u_1, u'_1, u'_2\}$ spans four edges. This means that we can apply Lemma 2.2 taking \bar{A} as the core, and u_0 and u'_1 being its “distinguished” vertices with their neighborhoods also sitting in the core. One can check that all conditions are satisfied, and with our new values of $x \geq x_0 = 2$, $x' \geq x'_0 = 2$ and $e = 4$, we get that G has at least $3n - 8 - 0 + 4 + 2 = 3n - 2$ edges. This contradiction finishes the proof of the theorem. \square

5 Concluding remarks

Although we could slightly improve the error term in Theorem 2.5, it seems that more ideas are needed to prove the full conjecture. We also note that our methods can be used to provide an asymptotically tight estimate on the minimum number of edges in a $K_{s,t}$ -saturated unbalanced

bipartite graph (i.e., with parts of size m and n). Determining the precise value in this unbalanced case might be even more challenging, although we believe that a straightforward modification of the extremal construction from the balanced case is tight here as well.

Bipartite saturation results were generalized to the hypergraph setting in [1, 8], where G and H are assumed to be k -partite k -uniform hypergraphs, and G is H -saturated if any new hyperedge meeting one vertex from each color class creates a new copy of H . It would be interesting to extend our results to get an asymptotically tight bound for the unordered k -partite hypergraph saturation problem.

Acknowledgment. We would like to thank A. Shapira for bringing this problem to our attention.

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